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NUC TP 509

TIME EVOLVING SPECTRA

by

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Undersea Surveillance Department

April 1976

NAVAL UNDERSEA CENTER

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SAN DIEGO, CALIFORNIA

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NUC-TP-509	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Time Evolving Spectra		5. TYPE OF REPORT & PERIOD COVERED Rept. for Sep '74 - Sep 75,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) O. D. Grace		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Undersea Center San Diego, CA 92132		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NISC project R3651, element 658 06N
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Intelligence Support Center 4301 Suitland Rd Washington D.C. 20390		12. REPORT DATE April 1976
		13. NUMBER OF PAGES 50
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Time Evolving Spectra   Nonstationary Processes
Spectral Analysis   Spectral Estimation
Fourier Transform

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Conventional spectral analysis methods do not describe the distribution of signal power in both time and frequency. Time dependent Fourier transforms are defined to extend the conventional theory to include instantaneous energy and power spectra for the past and future signal. These spectra are combined to → next page

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define a generalized instantaneous power spectrum (GIPS) of the entire signal. The Sliding FFT method of calculating the GIPS is examined and the equivalence between the estimates for the real and analytic signal is shown. These methods are applied to nonstationary stochastic processes and the GIPS is shown as given by the Fourier transform of Loeve's generalized power spectrum of the process and to be the generalization of the Wiener-Kintchine theorem to nonstationary processes. It is also shown to represent the stationary process, the locally stationary process, and the deterministic process. In addition to the GIPS, i.e., the spectral mean, an expression for the spectral covariance, is given and the relationship between these functions and a GIPS estimate via an ensemble of Sliding FFT calculations is examined.

SUMMARY

Problem: Develop spectral relations for analyzing processes having time
evolving spectra.

Results: Power spectral relations for nonstationary processes have been
developed that generalize the Wiener-Kintchine relation for stationary
processes.

Recommendations: Extend this analysis to include the relations between
ensemble processing and sequential processing for nonstationary processes.

I. INTRODUCTION

Spectral decomposition by Fourier transformation is a useful method of analyzing the transformations that signals undergo during generation, transmission, and reception. However, these methods are only in a partial state of development since they are not capable of describing the time evolution of the energy distribution in frequency of even the simpler signals that are encountered in everyday experience. We investigate these deficiencies in some detail and formulate improved methods. These formulations relate and extend the work of a number of earlier investigators.[1-13]

Representing the signal, $s(t)$, by the Fourier transform relations

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \exp(-i \, \omega t) \, d\omega \tag{1a}$$

$$S(\omega) = \int_{-\infty}^{\infty} s(t) \exp(i \, \omega t) \, dt \tag{1b}$$

we note that the signal can be viewed as a superposition of sinusoids having a range of frequencies, phases, and amplitudes. This interpretation is given more substance by Parseval's relation

$$\int_{-\infty}^{\infty} s^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega, \tag{2}$$

in which the left member is the signal energy and the integrand of the right member gives the energy per Hz and is called the energy spectrum. The usefulness of this decomposition for analyzing signal transformations is well known and has been thoroughly explored in many publications. The

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narrowness of these concepts can be appreciated by considering the sound generated by, say, a symphony orchestra or, more simply, a siren. If these signals were analyzed by the above methods the resulting energy spectrum would not describe the observers feeling that the signal is varying in frequency and time simultaneously. This lack of time dependence in conventional spectral analysis methods can be eliminated by means of recently developed methods.

Following a brief discussion of signals and their power and energy relations, the approach to a time evolving spectra is presented via the running Fourier transform that was originated by Page[1] and extended by Levin.[2] We develop their definitions for the deterministic signal and relate the resulting generalized instantaneous power spectrum (GIPS) to the sliding FFT output. These concepts are then extended to the general case of nonstationary stochastic signals by a method similar to Lampard's.[3] It is shown that the resulting GIPS of a nonstationary process is given by the Fourier transform of its generalized power spectral density[4] and also includes the well-known stationary case,[5] i.e., the Wiener-Kintchine theorem, the locally stationary case,[6] and the deterministic case.[2,7] In addition, the spectral variance of the process is shown to be bounded by the double Fourier transform of the covariance of the signal covariance estimate and both the GIPS estimate and its variance are related to the Sliding FFT output.

An earlier approach to the problem of time-frequency representation of a signal was given by Gabor.[8] He represents the signal in terms of a double sum of elementary signals that are distributed in time and frequency. The

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coefficients in the sum are evaluated by an iterative method and these

coefficients represent the time-frequency spectrum. This method gives good

insight into the distribution of signal energy in time and frequency but is

not a rigorous definition of the time dependent spectrum because the

elementary signals overlap. Helstrom[9] discusses an analogous integral

representation having this same deficiency while Montgomery and Reed[10]

generalize this integral representation to include a broader class of

elementary signals. It will be seen that their generalization is equivalent

to the method of Page for the appropriate class of elementary signals.

II. DETERMINISTIC SIGNALS

A. Signals, Power and Energy

Deterministic signals can be represented in the real form

$$f(t) = x(t) \cos \omega_0 t + y(t) \sin \omega_0 t$$

$$= A(t) \cos(\omega_0 t - \phi(t)) \tag{3}$$

where it is convenient to assume that the quadrature components are low passband functions that are limited to the band $\pm W/2$ $(W/2 \leq f_0 = \omega_0/2\pi)$

$$x(t) = \frac{1}{2\pi} \int_{-\pi W}^{\pi W} X(\omega) \exp(-i \omega t) \, d\omega \tag{4a}$$

$$y(t) = \frac{1}{2\pi} \int_{-\pi W}^{\pi W} Y(\omega) \exp(-i \omega t) \, d\omega \tag{4b}$$

and the envelope and phase are given by, respectively,

$$A(t) = [x^2(t) + y^2(t)]^{1/2} \tag{5a}$$

$$\phi(t) = \arctan [y(t)/x(t)]. \tag{5b}$$

The Fourier transform of this signal can be expressed in terms of the quadrature component transforms

$$F(\omega) = \frac{1}{2} [X(\omega-\omega_0) + i \, Y(\omega-\omega_0)] + \frac{1}{2} [X(\omega+\omega_0) - i \, Y(\omega+\omega_0)] \tag{6}$$

where the first term vanishes for negative frequencies and the second term vanishes for positive frequencies.

The power of the signal is given by

$$P(t) = f^2(t)$$

$$= \frac{1}{2} A^2(t) + \frac{1}{2} [(x^2(t) - y^2(t)) \cos 2\omega_0 t$$

$$+ 2x(t)\, y(t) \sin 2\omega_0 t], \qquad (7)$$

i.e., if f is voltage then P is the power dissipated in a unit resistance. This equation shows that the power contains a part that varies slowly compared to the carrier frequency and a part that varies at twice the carrier frequency. It can also be shown that the contribution of the oscillatory term to the total energy vanishes

$$E = \int_{-\infty}^{\infty} P(t)\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} A^2(t)\, dt, \qquad (8)$$

i.e., the oscillatory integrals are shown to vanish by substituting from Eq. 4, integrating the delta functions, and noting that the products of the form $U_1(\omega)\, U_2(2\omega_0 \pm \omega)$ vanish. This relation shows that only the first term in Eq. 7 contributes to the total energy of the signal and is the effective power of the signal while the second term contributes nothing to the total energy of the signal and is the reactive power of the signal, i.e., that part of the energy flow that is stored in the reactive elements of the system or medium.

Since the reactive power does not contribute to the net flow of energy it will be convenient to ignore it in the following and to consider only the effective power of the signal.

Deterministic signals can also be represented by the complex analytic signal

$$s(t) = m(t) \exp(-i \omega_0 t), \qquad (9)$$

where the complex envelope is related to the quadrature components by the expression

$$m(t) = x(t) + i y(t). \qquad (10)$$

This representation is frequently useful because it simplifies many calculations. The real and imaginery parts of the analytic signal

$$f(t) = \text{Re } s(t) \qquad (11a)$$

$$\hat{f}(t) = \text{Im } s(t) \qquad (11b)$$

represent, respectively, the real signal and a quadrature signal that is phase shifted by $\pi/2$ radians. The envelope and phase of both signals are obtained from the complex envelope by the expressions

$$A(t) = |m(t)|^2 \qquad (12a)$$

$$\phi(t) = \text{arg } m(t). \qquad (12b)$$

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The Fourier transform of the quadrature signal is given by

$$\hat{F}(\omega) = \frac{1}{2} [Y(\omega-\omega_0) - i X(\omega-\omega_0)] + \frac{1}{2} [Y(\omega+\omega_0) + i X(\omega+\omega_0)]$$

$$= i \, \text{sgn}(\omega) \, F(\omega), \tag{13}$$

where

$$\text{sgn}(\omega) = \pm 1, \quad \omega \gtrless 0 \tag{14}$$

and the Fourier transform of the analytic signal is given by

$$S(\omega) = F(\omega) + i [i \, \text{sgn}(\omega) \, F(\omega)]$$

$$= 2 \, U(\omega) \, F(\omega), \tag{15}$$

where

$$U(\omega) = 1, \quad \omega > 0$$

$$= 0, \quad \omega < 0 \tag{16}$$

which is just twice the real signal transform for positive frequencies and zero for negative frequencies. This property of the analytic signal is frequently useful.

The real signal power can be expressed in terms of the analytic signal by

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$$P(t) = [(s(t) + s\text{*}(t))/2]^2$$

$$= \frac{1}{2} \left| s(t) \right|^2 + \frac{1}{4} (s^2(t) + s\text{*}^2(t)), \tag{17}$$

where the superscript asterisk denotes complex conjugate and the terms can be shown to represent, respectively, the effective power and the reactive power. Consequently, the effective power of the real signal can be obtained from the modulus of the analytic signal.

B.   Deterministic Spectra

The energy of a signal is distributed in both time and frequency and at any time t is a function of only the signal in the past since the future of the signal must be considered arbitrary. One approach to estimating this distribution is to pass the signal through a bank of bandpass filters. The impulse response of one such filter in the bank is $h(t) \cos \omega t$ and the filter output is

$$\tilde{g}^-(t, \omega) = \int_{-\infty}^{t} f(\tau) \, h(t-\tau) \, \cos[\omega(t-\tau)] \, d\tau. \tag{18}$$

For the bank of filters the collection of functions $\tilde{g}^-(t, \omega)$ provides a means of estimating the distribution of signal energy in time and frequency. We note that the filter bank is not necessarily a comb filter in which the individual filter bandwidths are nonoverlapping.

The effective power of each filter output can be found by a quadrature demodulation to obtain the envelope of the output. The quadrature components

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of $\tilde{g}^-(t, \omega)$ are found by multiplying Eq. 18 by 2 cos ωt and 2 sin ωt, (where the factors of 2 are required to preserve the output power) and the terms at frequency 2ω are eliminated by filtering

$$\tilde{x}^-(t, \omega) = \int_{-\infty}^{t} f(\tau) \ h(t-\tau) \ \cos \omega\tau \ d\tau \tag{19a}$$

$$\tilde{y}^-(t, \omega) = \int_{-\infty}^{t} f(\tau) \ h(t-\tau) \ \sin \omega\tau \ d\tau. \tag{19b}$$

The complex envelope of this output is defined by

$$\tilde{G}^-(t, \omega) = \tilde{x}^-(t, \omega) + i \ \tilde{y}^-(t, \omega) \tag{20}$$

and we obtain

$$\tilde{G}^-(t, \omega) = \int_{-\infty}^{t} f(\tau) \ h(t-\tau) \ \exp(i \ \omega\tau) \ d\tau. \tag{21}$$

If the real input signal were replaced by the analytic signal and the real filter were replaced by the analytic filter, $h(t) \exp(-i \ \omega\tau)$, then the output would immediately be found to be

$$\tilde{G}^-(t, \omega) = \int_{-\infty}^{t} s(\tau) \ h(t-\tau) \ \exp(i \ \omega\tau) \ d\tau, \tag{22}$$

where we have heterodyned by multiplying by $\exp(i \ \omega\tau)$. Since both Eq. 21 and Eq. 22 have the same form we can pursue this development in terms of the analytic signal or the real signal merely by a change of notation. We use the analytic signal notation except as noted.

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The squared envelope of the output is

$$\left| \tilde{G}^-(t, \omega) \right|^2 = \left| \int_{-\infty}^{t} s(\tau) \, h(t-\tau) \, \exp(i \, \omega\tau) \, d\tau \right|^2, \qquad (23)$$

and if this function is integrated over frequency

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \tilde{G}^-(t, \omega) \right|^2 d\omega = \int_{-\infty}^{t} s^2(\tau) \, h^2(t-\tau) \, d\tau \qquad (24)$$

we find that the result is the total energy stored in the filter at time t and that the function $\left| \tilde{G}^-(t, \omega) \right|^2$ gives the energy distribution in frequency of the filter output at time t, i.e., the instantaneous energy spectrum (IES).

The IES as defined above is dependent upon the signal and also upon the choice of filter because the impulse response selectively weights each portion of the past signal. We can define a spectrum that is dependent only upon the signal by choosing a filter whose impulse response is nondistorting, i.e., the unit step function

$$h(t) = U(t) = 1, \quad t > 0,$$

$$= 0, \quad t < 0, \qquad (25)$$

which gives equal weight to all portions of the past signal.

The complex envelope becomes

$$G^-(t, \omega) = \int_{-\infty}^{t} s(\tau) \, \exp(i \, \omega\tau) \, d\tau, \qquad (26)$$

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which is the same as the running Fourier transform defined by Page and which can be inverted to obtain the signal representation

$$\frac{1}{2} s(t^-) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^-(t, \omega) \exp(-i \omega t) \, d\omega. \tag{27}$$

The resulting IES is the running energy spectrum defined by Page

$$|G^-(t, \omega)|^2 = \left| \int_{-\infty}^{t} s(\tau) \exp(i \omega \tau) \, d\tau \right|^2 \tag{28}$$

and represents the distribution of energy in frequency due to the past signal

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |G^-(t, \omega)|^2 \, d\omega = \int_{-\infty}^{t} |s(\tau)|^{2^-} \, d\tau. \tag{29}$$

The case of the filter dependent IES, in which a decaying impulse response was used, has been discussed by Fano[11] and later by Schroeder and Atal[12] as a means of estimating the short time spectrum. Differentiating Eq. 29 we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |G^-(t, \omega)|^2 \, d\omega = |s(t)|^2, \tag{30}$$

which shows that the function

$$\rho^-(t, \omega) = \frac{\partial}{\partial t} |G^-(t, \omega)|^2 \tag{31}$$

is the instantaneous power spectrum (IPS) of the signal as defined by Page and which is uniquely defined by the signal.

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Expanding the IPS definition in Eq. 31 by means of Eq. 28 we obtain Page's result

$$\rho^-(t, \omega) = 2\mathrm{Re}[s(t) \exp(i \omega t) G^{-*}(t, \omega)].$$ (32)

By the previous arguments this expression represents the distribution of signal power in time and frequency. We note that, while this power distribution is always real, it may be negative which implies that the signal energy can be redistributed in frequency as time evolves. We also note that this relation is unobservable since any attempt to place a "time-frequency window" on the signal changes the preceding relations.

The IPS and the IES, as defined, are dependent only on the past signal but, following Levin, analogous functions can be defined for the future portion of the signal:

$$G^+(t, \omega) = \int_t^\infty s(\tau) \exp(i \omega \tau) d\tau$$ (33a)

$$\frac{1}{2} s(t^+) = \frac{1}{2\pi} \int_{-\infty}^\infty G^+(t, \omega) \exp(-i \omega \tau) d\omega$$ (33b)

$$|G^+(t, \omega)|^2 = \left| \int_t^\infty s(\tau) \exp(i \omega \tau) d\tau \right|^2$$ (33c)

$$\frac{1}{2\pi} \int_{-\infty}^\infty |G^+(t, \omega)|^2 d\omega = \int_t^\infty |s(\tau)|^2 d\tau$$ (33d)

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{\partial}{\partial t} |G^+(t, \omega)|^2 d\omega = -|s(t)|^2$$ (33e)

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$$\rho^+(t, \omega) = -\frac{\partial}{\partial t} \left| G^+(t, \omega) \right|^2 \tag{33f}$$

$$= 2\text{Re} \left[s(t) \exp(i \omega t) \, G^{+*}(t, \omega) \right]. \tag{33g}$$

The minus sign is introduced into the IPS definition in Eq. 33f because
of the minus sign in Eq. 33e. These functions are not directly measureable
but are useful because they can be used to define general relations that are
valid for all time and frequency. Prior to developing these general relations
we note that the past expressions $\left| G^- \right|^2$ and ρ^- are independent of the future
relations $\left| G^+ \right|^2$ and ρ^+. The physical argument for this conclusion appears
to be that the past expressions represent the energy and energy flow re-
quired to generate the signal up to time t, while the future expressions
represent the energy and energy flow required to terminate the signal after
time t. Since the past and future portions of the signal are necessarily
independent the past and future relations are independent, i.e., they do not
give the same distributions at time t.

The expressions for the past and future running Fourier transforms can
be added to obtain the signal Fourier transform

$$G^-(t, \omega) + G^+(t, \omega) = \int_{-\infty}^{\infty} s(\tau) \exp(i \omega \tau) \, d\tau$$

$$= G(\omega). \tag{34a}$$

Similarly, the sum of the inversion formulas gives the signal representation

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$$\frac{s(t^-) + s(t^+)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [G^-(t, \omega) + G^+(t, \omega)] \exp(-i \omega\tau) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \exp(-i \omega\tau) \, d\omega. \qquad (34b)$$

The sum of the past and future power relations vanish

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [\frac{\partial}{\partial t} |G^-(t, \omega)|^2 + \frac{\partial}{\partial t} |G^+(t, \omega)|^2] \, d\omega = 0, \qquad \textbf{(34c)}$$

which shows that the total energy flow into the past equals the total energy flow from the future; but we note that this does not imply that the IPS for past and future are equal and of opposite sign. The difference between the past and future power relations

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} [|G^-(t, \omega)|^2 - |G^+(t, \omega)|^2] \, d\omega = 2|s(t)|^2 \qquad (34d)$$

shows that we can define the generalized instantaneous power spectrum (GIPS)

$$\rho(t, \omega) = \frac{1}{2} \frac{\partial}{\partial t} [|G^-(t, \omega)|^2 - |G^+(t, \omega)|^2] \qquad (34e)$$

$$= \text{Re} [s(t) \exp(i \omega t) S*(\omega)], \qquad (34f)$$

which contains information concerning both the past and future of the signal. This spectrum represents the power distribution for all times and frequencies. The respective integrals of the GIPS

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$$\int_{-\infty}^{\infty} \rho(t, \omega) \, dt \;=\; |S(\omega)|^2 \tag{35a}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(t, \omega) \, d\omega \;=\; |s(t)|^2 \tag{35a}$$

$$\frac{1}{2\pi} \iint_{-\infty}^{\infty} \rho(t, \omega) \, d\omega \, dt \;=\; E \tag{35c}$$

give the energy density in frequency, the energy density in time, and the total energy. A convenient method of computing the GIPS is given in Section D.

C. An Example

The example that we consider is the sinusoidal signal

$$s(t) \;=\; \exp(-i \, \omega_0 t), \qquad 0 \leq t \leq T, \tag{36}$$

$$= \; 0 \;\; \text{other times.}$$

The running Fourier transform, IES and IPS for the past signal are readily found to be

$$G^-(t, \omega) \;=\; t \, \exp[i(\omega-\omega_0)t/2] \, \text{sinc}^2 \, [(\omega-\omega_0)t/2] \tag{37a}$$

$$|G^-(t, \omega)|^2 \;=\; t^2 \, \text{sinc}^2 \, [(\omega-\omega_0)t/2] \tag{37b}$$

$$\rho^-(t, \omega) \;=\; 2t \, \text{sinc}[(\omega-\omega_0)t], \tag{37c}$$

$$0 \leq t \leq T,$$

where

$$\text{sinc}(x) = \sin(x)/x. \tag{38}$$

We see that each function has a similar form and represents a ridge positioned at the carrier frequency that grows in amplitude and decreases in width with time. This shows that the bandwidth of a sinusoidal pulse is initially very wide but that it approaches the theoretical limit of zero as time increases. The running Fourier transform, IES and IPS of the future signal are similarly found to be

$$G^+(t, \omega) = T \exp[i(\omega-\omega_0)T/2] \text{ sinc}[(\omega-\omega_0)T/2]$$

$$- t \exp[i(\omega-\omega_0)t/2] \text{ sinc}[(\omega-\omega_0)t/2]. \tag{39a}$$

$$|G^+(t, \omega)|^2 = T^2 \text{ sinc}^2[(\omega-\omega_0)T/2] + t^2 \text{ sinc}^2[(\omega-\omega_0)t/2]$$

$$- 2tT \cos[(\omega-\omega_0)(T-t)/2]$$

$$x \text{ sinc}[(\omega-\omega_0)T/2] \text{ sinc}[(\omega-\omega_0)t/2] \tag{39b}$$

$$\rho^+(t, \omega) = 2T \cos[(\omega-\omega_0)(T/2-t)] \text{ sinc}[(\omega-\omega_0)T/2]$$

$$-2t \cos[(\omega-\omega_0)t/2] \text{ sinc}[(\omega-\omega_0)t/2], \tag{39c}$$

$$0 \leq t \leq T.$$

Combining the IPS of the past and future signal gives the GIPS of the analytic sinusoid

$$\rho(t, \omega) = T \cos'[(\omega-\omega_0)(T/2-t)] \, \text{sinc}[(\omega-\omega_0)T/2], \qquad (40)$$

$$0 \le t \le T.$$

This function has the form of a ridge positioned at the carrier frequency but it has constant amplitude and the spectrum width is nominally constant.

D.   Calculation of the GIPS By the Sliding FFT

A common method of calculating the GIPS is to compute the power spectra of blocks of data windowed sequentially from the signal. The resulting power spectra are displayed sequentially and are assumed to represent cross sections through the GIPS surface. This calculation is normally effected by the fast Fourier transform algorithm and the result of the calculation is referred to here as the Sliding FFT. We will determine the relationship between the Sliding FFT and the GIPS.

The window to be applied to the data is assumed to be real and even and is represented by the Fourier transform

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\omega) \exp(-i \, \omega t) \, d\omega \qquad (41a)$$

$$P(\omega) = \int_{-\infty}^{\infty} p(t) \exp(i \, \omega t) \, dt. \qquad (41b)$$

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Extracting the block of data from the interval $t - T/2 \leq t' \leq t + T/2$ its power spectrum estimate at time t is

$$\tilde{\rho}(t, \omega) = \left| F(t, \omega) \right|^2$$

$$= \left| \int_{-\infty}^{\infty} dt' \, p(t') \, f(t'+t) \, \exp(i\, \omega t') \right|^2, \tag{42}$$

where we are assuming that the Sliding FFT is applied to the real signal. Repeating this calculation for a sequence of times t gives the Sliding FFT estimate of the GIPS.

The relationship between the Sliding FFT and the GIPS can be determined by transforming Eq. 42 by substituting the appropriate Fourier integral relations to obtain

$$\tilde{\rho}(t, \omega) = \frac{Re}{2\pi} \iint_{-\infty}^{\infty} dt' \, d\omega' \, [P_p^*(t'-t, \omega'-\omega) \, P_f(t', \omega')], \tag{43a}$$

where

$$P_p(t, \omega) = p(t) \, \exp(i\, \omega t) \, P^*(\omega) \tag{43b}$$

$$P_f(t, \omega) = f(t) \, \exp(i\, \omega t) \, F^*(\omega). \tag{43c}$$

The convolution in Eq. 43a shows that the Sliding FFT produces a smoothed estimate of the GIPS in which the smoothing window is the GIPS of the data

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window. An estimate of the effects of the smoothing can be found by estimating the size of the t-ω cell on which the GIPS of the window is significant.

The location of the cell is given by the weighted averages

$$\bar{t} = \frac{Re}{2\pi E} \iint_{-\infty}^{\infty} dt' \, d\omega' \, t' \, P_p(t'-t, \omega'-\omega) \tag{44a}$$

$$= t$$

$$\bar{\omega} = \frac{Re}{2\pi E} \iint_{-\infty}^{\infty} dt' \, d\omega' \, \omega' \, P_p(t'-t, \omega'-\omega) \tag{44b}$$

$$= \omega.$$

The size of the cell is given by $[\overline{(\omega-\bar{\omega})^2} \; \overline{(t-\bar{t})^2}]^{1/2}$ where

$$\overline{(\omega-\bar{\omega})^2} = \frac{Re}{2\pi E} \iint_{-\infty}^{\infty} dt' \, d\omega' \, (\omega'-\bar{\omega})^2 \, P_p(t'-t, \omega'-\omega)$$

$$= - \int_{-\infty}^{\infty} dt' \, p(t') \, \ddot{p}(t')/E$$

$$= \int_{-\infty}^{\infty} dt' \, |\dot{p}(t')|^2/E \tag{45a}$$

$$\overline{(t-\bar{t})^2} = \frac{Re}{2\pi E} \iint_{-\infty}^{\infty} dt' \, d\omega' \, (t'-\bar{t})^2 \, P_p(t'-t, \omega'-\omega)$$

$$= \int_{-\infty}^{\infty} dt' \, t'^2 |p(t')|^2/E, \tag{45b}$$

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where the dots in Eq. 45a denote derivatives. The integrals can be evaluated from the relation[13]

$$\int_{-\infty}^{\infty} dt´\, t´\, p(t´)\, \dot{p}(t´) \;=\; -[\; \int_{-\infty}^{\infty} dt´\, |p(t´)|^2 + \int_{-\infty}^{\infty} dt´\, t´\, p(t´)\, \dot{p}(t)\;]$$

$$= \; -E/2. \tag{46}$$

Transposing and squaring we obtain the bound

$$0 \;=\; [1/2 + \int_{-\infty}^{\infty} dt´\, t´\, p(t´)\, \dot{p}(t´)/E]^2$$

$$= \; (\int_{-\infty}^{\infty} dt´\, t´\, p(t´)\, \dot{p}(t´)/E)^2 - 1/4$$

$$\leq \frac{1}{E} \int_{-\infty}^{\infty} dt´\, t´^2 |p(t´)|^2 \; \frac{1}{E} \int_{-\infty}^{\infty} dt´\, |\dot{p}(t´)|^2 - 1/4. \tag{47}$$

Substituting from Eq. 45 we obtain the cell size expression

$$[\overline{(\omega-\overline{\omega})^2}\;\overline{(t-\overline{t})^2}]^{1/2} \;\geq\; 1/2. \tag{48}$$

We abbreviate this $\Delta\omega\Delta t \geq 1/2$, where Δt corresponds to the window length and $\Delta\omega$ corresponds to the spectral resolution. We note that the fast Fourier transform does not achieve the minimum resolution $\Delta\omega = 1/2\Delta t$.

For the minimum cell size (which can be obtained with the gaussian window, $\exp(-\alpha t^2)$), the Sliding FFT is approximated by

$$\tilde{\rho}(t, \omega) = \frac{\text{Re}}{2\pi} \int_{t-\Delta t}^{t+\Delta t} \int_{\omega-\Delta\omega}^{\omega+\Delta\omega} dt'\, d\omega'\, f(t')\, \exp(i\, \omega'\, t')\, F^*(\omega'), \tag{49}$$

where $\Delta t \Delta \omega = 1/2$. If the signal modulation and the Fourier transform are slowly varying over the domain of the window they can be approximated as

$$m(t') = m(t)\, \exp[-i\, \omega_t (t'-t)] \tag{50a}$$

$$F(\omega') = F(\omega)\, \exp[-i\, (\omega'-\omega)\, t_\omega], \tag{50b}$$

where ω_t and t_ω are the mean slopes of the phases of the modulation and Fourier transform over the cell. The instantaneous frequency of the signal at time t is $\omega_0 + \omega_t$ and the time shift of the Fourier component at the frequency ω is t_ω. Equation 49 becomes

$$\tilde{\rho}(t, \omega) = \frac{\text{Re}}{4\pi} \left\{ m(t)\, F^*(\omega) \int_{t-\Delta t}^{t+\Delta t} \int_{\omega-\Delta\omega}^{\omega+\Delta\omega} dt'\, d\omega'\, \exp[i\, (\omega'-\omega_0)t' \right.$$

$$-i\, \omega_t (t'-t)\, -i\, (\omega'-\omega)\, t_\omega]$$

$$+ m^*(t)\, F^*(\omega) \int_{t-\Delta t}^{t+\Delta t} \int_{\omega-\Delta\omega}^{\omega+\Delta\omega} dt'\, d\omega'\, \exp[i\, (\omega'+\omega_0)t'$$

$$\left. + i\, \omega_t (t'-t)\, -i\, (\omega'-\omega)t_\omega] \right\}, \tag{51}$$

and we note that the first term represents the Sliding FFT applied to the analytic signal while the sum of both terms represents the Sliding FFT applied to the real signal. The integrals in the expression can be transformed by the variable changes $t' \to t+t'$, $\omega' \to \omega+\omega'$ to obtain

$$I^{\pm} = \int_{t-\Delta t}^{t+\Delta t} \int_{\omega-\Delta\omega}^{\omega+\Delta\omega} dt'\, d\omega'\, \exp[i\,(\omega' \pm \omega_0)t' \pm i\,\omega_t(t'-t) - i\,(\omega'-\omega)t_\omega]$$

$$= \int_{-\Delta t}^{\Delta t} \int_{-\Delta\omega}^{\Delta\omega} dt'\, d\omega'\, \exp[i\,(\omega\pm\omega_0+\omega')\,(t+t') \pm i\,\omega_t\,t' - i\,\omega t_\omega]\ . \quad (52)$$

Expanding the product in the exponent and replacing the factor containing $\omega'\, t'$ by its Taylor series (for our purposes it is sufficient to retain just the first term) we obtain

$$I^{\pm} = \exp[i\,(\omega\pm\omega_0)t] \int_{-\Delta t}^{\Delta t} dt'\, \exp[i\,(\omega\pm\omega_0\pm\omega_t)\,t']$$

$$\times \int_{-\Delta\omega}^{\Delta\omega} d\omega'\, \exp[i\,\omega'(t-t_\omega)]$$

$$= 2\,\exp[i\,(\omega\pm\omega_0)t]\, \text{sinc}[(\omega\pm\omega_0\pm\omega_t)\Delta t]\, \text{sinc}[\Delta\omega(t-t_\omega)]. \quad (53)$$

With these expressions the Sliding FFT becomes

$$\tilde\rho(t,\omega) = \frac{\text{Re}}{2\pi}\left\{ [s(t)\,\text{sinc}[(\omega-\omega_0-\omega_t)\Delta t] + s^*(t)\,\text{sinc}[(\omega+\omega_0+\omega_t)\Delta t] \right.$$

$$\left. \times \exp(i\,\omega t)\,F^*(\omega)\,\text{sinc}[\Delta\omega(t-t_\omega)] \right\}. \quad (54)$$

In this expression the first term dominates on the positive band of frequencies and the second term dominates on the negative band of frequencies. Restricting

ourselves to just the positive band of frequencies we obtain the Sliding FFT
expression

$$\tilde{\rho}(t, \omega) = \frac{Re}{2\pi} [s(t) \exp(i \omega t) F^*(\omega)]$$

$$\times \operatorname{sinc}[(\omega-\omega_0-\omega_t)\Delta t] \operatorname{sinc}[\Delta\omega(t-t_\omega)], \tag{55}$$

which is a faithful reproduction of the GIPS in the region of dominant signal
activity $(\omega = \omega_0 + \omega_t, t = t_\omega)$ but which is distorted elsewhere. This
expression shows that the Sliding FFT of the real signal has approximately
the same form as the Sliding FFT of the analytic signal. Consequently, it
follows that a theoretically derived GIPS, based upon the analytic signal,
can be compared directly to the Sliding FFT of the real signal. This is an
important result since analyses involving the analytic signal are generally
simpler than those involving the real signal while the Sliding FFT is more
easily applied to the real signal.

III. STOCHASTIC PROCESSES

A. Harmonizability Theorem

A more general class of signal than the deterministic signal is the class of stochastic signals. We consider the class of harmonizable stochastic signals of Loeve[4], i.e., second order random functions $s(t)$ are harmonizable if there exists a second order spectral decomposition of $s(t)$, denoted $g(\omega)$, such that

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\,\omega t)\,dg(\omega). \tag{56}$$

The covariances of these functions are

$$\Gamma(t, t') = E[s(t)\,s^*(t')] \tag{57a}$$

$$\gamma(\omega, \omega') = E[g(\omega)\,g^*(\omega')] \tag{57b}$$

and we note that the time dependent effective power of the process is given by $\Gamma(t, t)$. By Loeve's harmonizability theorem we obtain the covariance relation

$$\Gamma(t, t') = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \exp[-i\,(\omega t - \omega'\,t')]\,dd'\,\gamma(\omega, \omega') \ . \tag{58}$$

These Fourier-Stieltjes integrals can be simplified to Riemann integrals by assuming mean square differentiability of the spectral decomposition of $s(t)$

$$dg(\omega) = G(\omega)\,d\omega, \tag{59}$$

and the signal representation and its inverse formula become

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\,\omega t)\, G(\omega)\, d\omega \tag{60a}$$

$$G(\omega) = \int_{-\infty}^{\infty} \exp(i\,\omega t)\, s(t)\, dt, \tag{60b}$$

where $G(\omega)$ is the random Fourier transform of the signal. Under the mean square differentiability assumption the integrator in the covariance relation becomes

$$dd'\,\gamma(\omega,\,\omega') = E[dg(\omega)\, d'\, g*(\omega')]$$

$$= E[G(\omega)\, G*(\omega')]\, d\omega\, d\omega'$$

$$= \Psi(\omega,\,\omega')\, d\omega\, d\omega', \tag{61}$$

where $\Psi(\omega,\,\omega')$ is Loeve's "generalized power spectral density" of the process and the covariance relation and its inverse relation become

$$\Gamma(t,\,t') = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \exp[-i\,(\omega t - \omega' t')]\, \Psi(\omega,\,\omega')\, d\omega\, d\omega' \tag{62a}$$

$$\Psi(\omega,\,\omega') = \iint_{-\infty}^{\infty} \exp[i\,(\omega t - \omega' t')]\, \Gamma(t,\,t')\, dt\, dt'. \tag{62b}$$

The covariance relations in Eq. 61 show that the spectral components in disjoint frequency bands are correlated, which in the context of time evolving spectra appears to imply that on the average energy may be transported between these bands; i.e., that the ensemble power spectrum

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is time evolving. The time dependence is not apparent in Eq. 62 but in the following we develop this viewpoint and show that the GIPS of the process is given by the Fourier transform of the covariance function, i.e., the generalized Wiener-Kintchine relation, or the Fourier transform of Loeve's generalized power spectral density. Two cases of special interest will be considered prior to this development.

B. Two Special Cases

1. Wide Sense Stationary Processes

For the classical case of wide sense stationary processes[5] the covariance function depends only on the time differences and not on the absolute time

$$\Gamma(t, t') = \Gamma(t-t'). \tag{63}$$

The generalized power spectral density in Eq. 62b becomes

$$
\begin{aligned}
\Psi(\omega, \omega') &= \iint_{-\infty}^{\infty} \exp[i\,(\omega t - \omega'\,t')]\,\Gamma(t-t')\,dt\,dt' \\
&= \int_{-\infty}^{\infty} \exp[i\,(\omega-\omega')t']\,dt' \int_{-\infty}^{\infty} \exp(i\,\omega\tau)\,\Gamma(\tau)\,d\tau \\
&= 2\pi\,\delta(\omega-\omega')\,\Psi(\omega), \tag{64}
\end{aligned}
$$

where the first integral is the delta function and the covariance representation and its inverse formula are given by

$$\Gamma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\,\omega t)\,\Psi(\omega)\,d\omega \tag{65a}$$

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$$\Psi(\omega) = \int_{-\infty}^{\infty} \exp(i\,\omega t)\,\Gamma(t)\,dt. \tag{65b}$$

This is the well-known Wiener-Kintchine result for stationary processes. Equation 64 shows that stationarity is equivalent to assuming that the spectral power is orthogonal, i.e.,

$$E[dg(\omega)\,d'\,g^*(\omega')] = 2\pi\,\delta(\omega-\omega')\,\Psi(\omega)\,d\omega\,d\omega', \tag{66}$$

which implies that the power in disjoint frequency bands is statistically uncorrelated. In the context of time evolving spectra this seems to imply that on the average energy is not transported between bands and consequently the power spectrum is constant in time. It is also readily seen that the time dependent power, $\Gamma(0)$, is constant.

2. Locally Wide Sense Stationary Processes

Silverman[6] has extended the concept of wide sense stationarity by defining locally wide sense stationary processes, i.e., processes that have a covariance function of the form

$$\Gamma(t,\,t') = \Gamma_1\left(\frac{t+t'}{2}\right)\,\Gamma_2(t-t'), \tag{67}$$

where Γ_1 is a nonnegative function and Γ_2 is a stationary covariance. We note that if Γ_2 is normalized to unity then Γ_1 is the time varying effective power

$$\Gamma(t,\,t) = \Gamma_1(t). \tag{68}$$

Substituting in Eq. 62b we obtain for the generalized power spectral density

$$\Psi(\omega, \omega') = \iint_{-\infty}^{\infty} \exp[i\,(\omega t - \omega' t')]\, \Gamma_1(\frac{t+t'}{2})\, \Gamma_2(t-t')\, dt\, dt'$$

$$= \iint_{-\infty}^{\infty} \exp[i\,(\omega - \omega')\,(\frac{t+t'}{2}) + (\frac{\omega+\omega'}{2})\,(t-t')]$$

$$\times\, \Gamma_1(\frac{t+t'}{2})\, \Gamma_2(t-t')\, dt\, dt'. \tag{69}$$

Substituting the variables $u = (t+t')/2$ and $v = (t-t')$, we obtain

$$\Psi(\omega, \omega') = \Psi_1(\omega-\omega')\, \Psi_2(\frac{\omega+\omega'}{2}), \tag{70}$$

where

$$\Psi_1(\omega) = \int_{-\infty}^{\infty} \exp(i\,\omega t)\, \Gamma_1(t)\, dt \tag{71a}$$

$$\Psi_2(\omega) = \int_{-\infty}^{\infty} \exp(i\,\omega t)\, \Gamma_2(t)\, dt \tag{71b}$$

$$\Gamma_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\,\omega t)\, \Psi_1(\omega)\, d\omega \tag{71c}$$

$$\Gamma_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\,\omega t)\, \Psi_2(\omega)\, d\omega, \tag{71d}$$

which shows that a process with a locally stationary covariance has a
generalized power spectral density that is also a locally stationary
covariance.

We note that the locally stationary case reduces to the stationary case
if the time varying power is constant, say unity,

$$\Gamma_1 = 1. \tag{72a}$$

The spectral density of Γ_1 is found from Eq. 71a to be

$$\Psi_1(\omega) = 2\pi \, \delta(\omega), \tag{72b}$$

and we obtain from Eq. 67 and Eq. 70

$$\Gamma(t, t') = \Gamma_2(t-t') \tag{73a}$$

$$\Psi(\omega, \omega') = 2\pi \, \delta(\omega-\omega') \, \Psi_2(\omega) \tag{73b}$$

which are the same as for the stationary case in Eq. 63 and Eq. 64.

In the next section we derive the GIPS for a nonstationary second order stochastic process and show that it is given by the Fourier transforms of both the covariance function and the generalized power spectral density. Its relationship to the special cases of the stationary process, the locally stationary process, and the deterministic process is developed and we also obtain an expression for the variance of the ensemble of spectra.

C.   Nonstationary Processes

1.   Spectral Mean

The harmonizability expressions in Eq. 60 can be resolved into past and future relations and Page's IES can be extended directly to stochastic signals

$$\left| G^-(t, \omega) \right|^2 = \left| \int_{-\infty}^{t} s(t') \exp(i \, \omega t') \, dt' \right|^2 . \tag{74}$$

This function has the expected value

$$E\left|G^-(t, \omega)\right|^2 = \iint_{-\infty}^{t} \Gamma(\tau_1, \tau_2) \exp[i \omega(\tau_1-\tau_2)] \, d\tau_1 \, d\tau_2 , \qquad (75)$$

which is similar to Lampard's[3] joint energy function but we define the signal covariance function differently.

Separating these integrals and making the appropriate variable changes we obtain

$$E\left|G^-(t, \omega)\right|^2 = \int_{-\infty}^{t} d\tau_1 \int_{\tau_1}^{t} d\tau_2 \exp[i \omega(\tau_1-\tau_2)] \, \Gamma(\tau_1, \tau_2)$$

$$+ \int_{-\infty}^{t} d\tau_2 \int_{\tau_2}^{t} d\tau_1 \exp[i \omega(\tau_1-\tau_2)] \, \Gamma(\tau_1, \tau_2)$$

$$= 2 \, \mathrm{Re} \int_{-\infty}^{t} d\tau_1 \int_{0}^{t-\tau_1} d\tau \, \Gamma(\tau_1, \tau_1+\tau) \exp(-i \omega\tau).$$
$$(76)$$

This expression gives the mean energy distribution in frequency at time t, i.e., the integral of this function over frequency gives the energy of the past signal

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} E\left|G^-(t, \omega)\right|^2 d\omega = \frac{\mathrm{Re}}{\pi} \int_{-\infty}^{t} d\tau_1 \int_{0}^{t-\tau_1} d\tau \, \Gamma(\tau_1, \tau_1+\tau)$$

$$\times \int_{-\infty}^{\infty} d\omega \exp(-i \omega\tau)$$

$$= E^-(t). \qquad (77)$$

In this calculation we note that a factor of 1/2 is obtained because the "τ" integration is over just 1/2 the support of the delta function.

Following Page we define the IPS to be the time derivative of the IES

$$\rho^-(t, \omega) = \frac{\partial}{\partial t} E|G^-(t, \omega)|^2 \quad . \tag{78}$$

Substituting the IES from Eq. 76 into the IPS definition we obtain

$$\rho^-(t, \omega) = 2 \text{ Re} \int_{-\infty}^{0} d\tau_1 \, \Gamma(t+\tau_1, t) \exp(i \omega\tau_1), \tag{79}$$

which shows that the IPS of the past signal is the real part of the Fourier transform of the covariance of the past signal. The IPS inversion formula is found to be

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \rho^-(t, \omega) \exp(-i \omega\tau) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{0} d\tau_1$$

$$\times \left[\int_{-\infty}^{\infty} \exp[i \omega(\tau_1-\tau)] \, d\omega \, \Gamma(t+\tau_1, t) \right.$$

$$\left. + \int_{-\infty}^{\infty} \exp[-i \omega(\tau_1+\tau)] \, d\omega \, \Gamma^*(t+\tau_1, t) \right]$$

$$= \Gamma(t+\tau, t), \quad \tau < 0,$$

$$= \Gamma(t, t), \quad \tau = 0,$$

$$= \Gamma^*(t-\tau, t), \quad \tau > 0, \tag{80}$$

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where only the first or second term of the right member contributes for $\tau \gtrless 0$ while for $\tau = 0$ both terms contribute but both integrations are over 1/2 the support of the delta function. These Fourier transform relations between the IPS and the signal covariance in Eq. 79 and Eq. 80 are seen to be related to the Wiener-Kintchine relations for stationary processes. The above expressions differ though because they are time dependent and because they contain only information concerning the past signal.

Analogous expressions can be developed for the future part of the signal. The IES is given by

$$E|G^+(t, \omega)|^2 = \iint_t^\infty \Gamma(\tau_1, \tau_2) \exp[i \omega(\tau_1 - \tau_2)] \, d\tau_1, \, d\tau_2 \tag{81}$$

and manipulating as before we obtain

$$E|G^+(t, \omega)|^2 = 2 \, \text{Re} \int_t^\infty d\tau_1 \int_{t-\tau_1}^0 d\tau \, \Gamma(\tau_1, \tau_1+\tau) \exp(-i \omega\tau). \tag{82}$$

This expression gives the mean energy distribution in frequency at time t, i.e., the integral of this function over frequency gives the energy of the future signal

$$\frac{1}{2\pi} \int_{-\infty}^\infty E|G^+(t, \omega)|^2 \, d\omega = E^+(t). \tag{83}$$

The IPS is defined to be

$$\rho^+(t, \omega) = -\frac{\partial}{\partial t} E|G^+(t, \omega)|^2, \tag{84}$$

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and substituting from Eq. 82 we obtain

$$\rho^+(t, \omega) = 2 \, \text{Re} \int_0^\infty d\tau_1 \, \Gamma(t+\tau_1, t) \, \exp(i \, \omega\tau_1). \tag{85}$$

The IPS inversion formula is found to be

$$\frac{1}{2\pi} \int_{-\infty}^\infty \rho^+(t, \omega) \exp(-i \, \omega\tau) \, d\omega = \frac{1}{2\pi} \int_0^\infty d\tau_1$$

$$\times \left[\int_{-\infty}^\infty \exp[i \, \omega(\tau_1-\tau)] \, d\omega \, \Gamma(t+\tau_1, t) \right.$$

$$\left. + \int_{-\infty}^\infty \exp[-i \, \omega(\tau_1+\tau)] \, d\omega \, \Gamma^*(t+\tau_1, t) \right]$$

$$= \Gamma(t+\tau, t), \quad \tau > 0,$$

$$= \Gamma(t, t), \quad \tau = 0,$$

$$= \Gamma^*(t-\tau, t), \quad \tau < 0. \tag{86}$$

Analogous to the expressions for the past signal these Fourier transform relations between the IPS and the signal covariance are related to the Wiener-Kintchine relation for stationary processes.

The above expressions for the past and future of the signal can be combined to obtain results that relate both the past and the future. The sum of the past and the future energy flow relations can be obtained from Eq. 77 and Eq. 83 and vanish

$$\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t} E |G^-(t, \omega)|^2 + \frac{\partial}{\partial t} E |G^+(t, \omega)|^2 \right] d\omega = 0, \qquad (87)$$

which shows that the energy flow from the future equals the energy flow into the past. The difference of the past and future energy flow relations can be obtained from Eq. 80 and Eq. 86:

$$\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t} E |G^-(t, \omega)|^2 - \frac{\partial}{\partial t} E |G^+(t, \omega)|^2 \right] d\omega = 2 \Gamma(t, t); \qquad (88)$$

this is twice the signal power and shows that the GIPS can be defined as

$$\rho(t, \omega) = \frac{1}{2} \frac{\partial}{\partial t} [E |G^-(t, \omega)|^2 - E |G^+(t, \omega)|^2]. \qquad (89)$$

Substituting the IPS from Eq. 79 and Eq. 85 we obtain

$$\rho(t, \omega) = \text{Re} \int_{-\infty}^{\infty} d\tau \, \Gamma(t+\tau, t) \exp(i \, \omega\tau), \qquad (90)$$

which shows that the GIPS is given by the Fourier transform of the covariance function. This result is the Wiener-Kintchine relation generalized to the case of nonstationary processes. The inverse relations are given in Eq. 80 and Eq. 86. Inspection shows that the inverse relations do not give a covariance that is necessarily even in the time shift parameter, τ, which of course must be the case for a nonstationary process. This is in contrast to Lampard's[14] result. In addition we can relate the GIPS to Loeve's generalized power spectral density for nonstationary second order random processes by substituting directly from Eq. 62a:

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$$\rho(t, \omega) \;=\; \frac{Re}{2\pi} \int_{-\infty}^{\infty} d\omega' \; \Psi(\omega, \omega') \; \exp[i \, (\omega'-\omega)t]. \tag{91}$$

We find that the GIPS is given by the Fourier transform of the generalized power spectral density. In the remainder of this section we show that the GIPS is a general result that represents a number of special interest cases.

For the case of stationary processes the generalized power spectral density is orthogonal on disjoint bands as shown in Eq. 64 and the GIPS in Eq. 91 becomes

$$\rho(t, \omega) \;=\; \Psi(\omega). \tag{92}$$

The covariance function is a function of time differences only as in Eq. 63 and the covariance in Eq. 90 becomes

$$\Gamma(t+\tau, \, t) \;=\; E[s(t+\tau) \; s*(t)]$$

$$=\; \Gamma(t). \tag{93}$$

With Eq. 92 and Eq. 93 the GIPS in Eq. 90 becomes

$$\Psi(\omega) \;=\; Re \int_{-\infty}^{\infty} d\tau \; \Gamma(\tau) \; \exp(i \, \omega\tau). \tag{94a}$$

This expression is readily inverted to obtain the covariance representation

$$\Gamma(\tau) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \Psi(\omega) \; \exp(-i \, \omega\tau), \tag{94b}$$

and this Fourier transform pair is the Wiener-Kintchine theorem as in Eq. 65.

For the case of locally stationary processes the generalized power spectral density separates into a locally stationary covariance as in Eq. 70 and the GIPS in Eq. 91 becomes

$$\rho(t, \omega) = \frac{\text{Re}}{2\pi} \int_{-\infty}^{\infty} d\omega' \, \Psi_1(\omega-\omega') \, \Psi_2(\frac{\omega+\omega'}{2}) \, \exp[i \, (\omega'-\omega)t]$$

$$= \text{Re} \int_{-\infty}^{\infty} dt' \, \Gamma_1(t+t'/2) \, \Gamma_2(t') \, \exp(i \, \omega t'), \tag{95}$$

where we have substituted from Eq. 71a and Eq. 71b. This last result could have been obtained directly from the GIPS expression in Eq. 90 by using the locally stationary covariance definition in Eq. 67.

For the case of deterministic signals the covariance function has the simple form

$$\Gamma(t+\tau, t) = s(t+\tau)s*(t), \tag{96}$$

and the GIPS in Eq. 90 becomes

$$\rho(t, \omega) = \text{Re} \int_{-\infty}^{\infty} d\tau \, s(t+\tau) \, s*(t) \, \exp(i \, \omega\tau)$$

$$= \text{Re} \, s(t) \, \exp(i \, \omega t) \, S*(\omega), \tag{97}$$

which is the same as we obtained in Eq. 34f. This result shows that our apparent conflict in terminology is only apparent, i.e., we have been using the term GIPS for both the spectrum of a deterministic signal and the mean spectrum for nonstationary processes. The above calculation illustrates that

both usages are correct and that the deterministic signal is just a special case of the nonstationary random process.

2. Spectral Variance

The GIPS is the spectral mean of a nonstationary random process. Since an estimate of the GIPS is more easily obtained if the variance of the estimate is small, it is important to have a theoretical expression for computing this quantity. In the following we show that the spectral covariance can be bounded by the double Fourier transform of the covariance of the estimate of the signal covariance. The spectral variance is readily obtained from this result.

The covariance of the GIPS is given by

$$
\Delta^2 \rho(t,t',\omega) = \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} E\{[(|G^-(t, \omega)|^2 - |G^+(t, \omega)|^2)
$$

$$
-(E|G^-(t, \omega)|^2 - E|G^+(t, \omega)|^2)]
$$

$$
\times [(|G^-(t', \omega)|^2 - |G^+(t', \omega)|^2)
$$

$$
-(E|G^-(t', \omega)|^2 - E|G^+(t', \omega)|^2)]\}
$$

$$
= \Delta^2 \rho^= (t,t',\omega) + \Delta^2 \rho^\pm(t,t',\omega) + \Delta^2 \rho^{\overline{\mp}}(t,t',\omega),
$$

$$
+ \Delta^2 \rho^{\ddagger}(t,t',\omega), \tag{98}
$$

where we define the covariances as

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$$\Delta^2 \rho^{=}(t,t',\omega) \;=\; \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} E[(|G^-(t,\omega)|^2 - E|G^-(t,\omega)|^2)$$

$$\times \; (|G^-(t',\omega)|^2 - E|G^-(t',\omega)|^2)] \qquad \text{(99a)}$$

$$\Delta^2 \rho^{\pm}(t,t',\omega) \;=\; \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} E[(|G^+(t,\omega)|^2 - E|G^+(t,\omega)|^2)$$

$$\times \; (|G^-(t',\omega)|^2 - E|G^-(t',\omega)|^2)] \qquad \text{(99b)}$$

$$\Delta^2 \rho^{\overline{+}}(t,t',\omega) \;=\; \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} E[(|G^-(t,\omega)|^2 - E|G^-(t',\omega)|^2)$$

$$\times \; (|G^+(t',\omega)|^2 - E|G^+(t',\omega)|^2)] \qquad \text{(99c)}$$

$$\Delta^2 \rho^{\overline{+}}(t,t',\omega) \;=\; \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} E[(|G^+(t,\omega)|^2 - E|G^+(t,\omega)|^2)$$

$$\times \; (|G^+(t',\omega)|^2 - E|G^+(t',\omega)|^2)] \qquad \text{(99d)}$$

The first of these terms is found from Eq. 74 and Eq. 75 to be

$$\Delta^2 \rho^{=}(t,t',\omega) \;=\; \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} E \left\{ \iint_{-\infty}^{t} [s(t_1)\, S^*(t_2) - \Gamma(t_1,t_2)] \right.$$

$$\times \; \exp(i\,\omega(t_1 - t_2))\; dt_1\, dt_2$$

$$\times \; \iint_{-\infty}^{t} [s^*(t_1')\, s(t_2') - \Gamma(t_1',t_2',)]$$

$$\left. \times \; \exp(-i\,\omega(t_1' - t_2'))\; dt_1'\, dt_2' \right\}, \qquad \text{(100)}$$

and the method of integration leading to Eq. 76 gives immediately

$$\Delta^2 \rho^{=}(t,t',\omega) = \frac{\partial^2}{\partial t \partial t'} E \left\{ \text{Re} \int_{-\infty}^{t} d\tau_1 \int_{0}^{t-\tau_1} d\tau \, [s(\tau_1) \, s*(\tau_1+\tau) - \Gamma(\tau_1,\tau_1+\tau)] \right.$$

$$\times \exp(-i\,\omega\tau) \, \text{Re} \int_{-\infty}^{t'} d\tau_1' \int_{0}^{t'-\tau_1'} d\tau'$$

$$\left. [s*(\tau_1')s(\tau_1'+\tau')-\Gamma*(\tau_1',\tau_1'+\tau')] \, \exp(i\,\omega\tau') \right\}$$

$$= E \left\{ \text{Re} \int_{-\infty}^{t} d\tau_1 \, [s(\tau_1) \, s*(t)-\Gamma(\tau_1, \, t)] \, \exp(-i\,\omega(t-\tau_1)) \right.$$

$$\left. \times \text{Re} \int_{-\infty}^{t'} d\tau_1' \, [s*(\tau_1')s(t')-\Gamma*(\tau_1', \, t')] \, \exp(i\,\omega(t'-\tau_1')) \right\}$$

$$= E \left\{ \text{Re} \int_{-\infty}^{0} d\tau_1 \, [s(t+\tau_1) \, s*(t)-\Gamma(t+\tau, \, t)] \, \exp(i\,\omega\tau_1) \right.$$

$$\left. \times \text{Re} \int_{-\infty}^{0} d\tau_1' \, [s*(t'+\tau_1') \, s(t')-\Gamma*(t'+\tau', \, t')] \, \exp(-i\,\omega\tau_1') \right\}$$

$$= E[(\text{Re} \int_{-\infty}^{0} d\tau \, I(t,\tau,\omega)) \, (\text{Re} \int_{-\infty}^{0} d\tau' \, I*(t',\tau',\omega))],$$

$$(101a)$$

where we define the function

$$I(t,\tau,\omega) = [s(t+\tau) \, s*(t) - \Gamma(t+\tau, \, t)] \, \exp(i\,\omega\tau).$$

By similar methods the remaining three terms are given by

$$\Delta^2 \rho^{\pm}(t,t',\omega) = E[(Re \int_0^\infty d\tau \, I(t,\tau,\omega))(Re \int_{-\infty}^0 d\tau' \, I*(t',\tau',\omega))] \quad (101b)$$

$$\Delta^2 \rho^{\overline{+}}(t,t',\omega) = E[(Re \int_{-\infty}^0 d\tau \, I(t,\tau,\omega))(Re \int_0^\infty d\tau' \, I*(t',\tau',\omega))] \quad (101c)$$

$$\Delta^2 \rho^{\ddagger}(t,t',\omega) = E[(Re \int_0^\infty d\tau \, I(t,\tau,\omega))(Re \int_0^\infty d\tau' \, I*(t',\tau',\omega))], \quad (101d)$$

where Eq. 81 and the method of integration leading to Eq. 82 is used.

Combining terms the GIPS covariance expression becomes

$$\Delta^2 \rho(t,t',\omega) = E\left\{ Re \int_{-\infty}^\infty d\tau \, [s(t+\tau) \, s*(t) - \Gamma(t+\tau, t)] \exp(i\,\omega\tau) \right.$$

$$\left. \times Re \int_{-\infty}^\infty d\tau' \, [s*(t'+\tau') \, s(t') - \Gamma*(t'+\tau',t')] \exp(-i\,\omega\tau') \right\},$$
$$(103)$$

which shows it explicitly to be the covariance of the ensemble of spectra. This expression can be expanded but it is more convenient to obtain a simple bound on the GIPS covariance which is given by

$$\Delta^2 \rho(t,t',\omega) \leq \int_{-\infty}^\infty d\tau \, d\tau' \, E\{[s(t+\tau) \, s*(t) - \Gamma(t+\tau, t)]$$

$$\times [s*(t'+\tau') \, s(t') - \Gamma*(t'+\tau',t')]\}$$

$$\times \exp(i\,\omega(\tau-\tau')). \quad (104)$$

This bound is the double Fourier transform of the covariance of the estimate of the signal covariance and is also the covariance of the phasor representation

of the GIPS. If the phasor representation is uniformly distributed about the mean phasor, then the variation of the random GIPS is expected to achieve the upper bound for at least some t, t´, and ω and the bound is a useful one. We note further that the spectral variance is given by the function $\Delta^2 \rho(t,t,\omega)$.

D. Estimation of the GIPS by the Sliding FFT

Frequently it will be more convenient to estimate the GIPS by means of the Sliding FFT rather than by means of the covariance function. Since we have previously seen that the Sliding FFT algorithm produces a smoothed version of the GIPS it is important to determine its effects on our estimate and its variance.

From Eq. 55 we find the estimate of the GIPS to be

$$\tilde{\rho}(t, \omega) = \frac{Re}{2\pi} E[s(t) \exp(i \omega t) F^*(\omega)] s(t, \omega), \qquad (105)$$

where we define the smoothing function as

$$s(t, \omega) = \text{sinc}[(\omega - \omega_0 - \omega_t)t] \, \text{sinc}[\Delta\omega(t - t_\omega)]. \qquad (106)$$

By previously used methods we obtain

$$\tilde{\rho}(t, \omega) = \left[\frac{Re}{2\pi} \int_{-\infty}^{\infty} dt´ \, \Gamma(t+t´, t) \exp(i \omega t´) \right] s(t, \omega), \qquad (107) \quad \cdot$$

which is a smoothed version of the GIPS.

The covariance of the estimate is given by

$$\Delta^2 \tilde{\rho}(t,t',\omega) = E \left\{ \left[\frac{Re}{2\pi} s(t) \exp(i \omega t) F^\star(\omega) \right. \right.$$

$$\left. - \frac{Re}{2\pi} \int_{-\infty}^{\infty} d\tau \ \Gamma(t+\tau, t) \exp(i \omega \tau) \right]$$

$$\times \left[\frac{Re}{2\pi} s^*(t') \exp(-i \omega t') F^\star(\omega) \right.$$

$$\left. \left. - \frac{Re}{2\pi} \int_{-\infty}^{\infty} d\tau' \ \Gamma^\star(t'+\tau', t') \exp(-i \omega \tau') \right] \right\}$$

$$\times s(t, \omega) s^\star(t', \omega)$$

$$= \frac{E}{(2\pi)^2} \left\{ \int_{-\infty}^{\infty} d\tau [s(t+\tau) \ s^\star(t) - \Gamma(t+\tau, t)] \exp(i \omega \tau) \right.$$

$$\left. \times \int_{-\infty}^{\infty} d\tau' \ [s^\star(t'+\tau') \ s(t') - \Gamma^\star(t'+\tau', t')] \exp(-i \omega \tau') \right\}$$

$$\times s(t, \omega) \ s^\star(t', \omega), \tag{108}$$

which is the same as the covariance expression in Eq. 103 with the exception of the smoothing factor introduced by the Sliding FFT. Consequently, we see that estimates of the GIPS obtained via the Sliding FFT will be distorted by the smoothing window in both its mean and variance.

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IV. SUMMARY AND CONCLUSIONS

We have seen that conventional spectral analysis methods by Fourier decomposition do not lead to results that are in agreement with the intuitive notion that signals evolve in frequency as well as in time. This deficiency is due to the infinite limits of integration on the Fourier integral and can be eliminated by defining a Fourier transform in which the integration limits cover just the past signal. This definition of a time evolving transform is used to compute the instantaneous energy and instantaneous power spectrum of the past signal and these functions are directly measureable in real time. These functions contain no information concerning the future signal but such functions can be readily defined from the Fourier transform in which the integration limits cover just the future signal. These future signal expressions, which can only be estimated after the event, contain no information concerning the past signal but by combining with the past signal expressions can be used to define the GIPS that contains information concerning both the past and the future signal.

The GIPS is usually calculated by means of the Sliding FFT algorithm or some variation thereof. The function generated by the Sliding FFT algorithm is found to be a smoothed version of the GIPS in which the smoothing is typically over a time-frequency domain of size $\Delta\omega \; \Delta t \gtrsim 1/2$. On the positive band of frequencies this function is an approximation of the GIPS of both the analytic signal and the real signal whether the algorithm is applied to the analytic signal or to the real signal. This shows that the result of a theoretical spectral analysis that is based upon the analytic signal can be approximately compared to the function generated by the Sliding FFT even though applied to the real signal. This is important because theoretical analyses based upon the

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analytic signal are generally simpler while the Sliding FFT is most conveniently
applied to the real signal.

The concepts derived for the deterministic signal are readily extended to
the case of nonstationary stochastic ensembles of signals. The GIPS of the
ensemble is found to be a generalization of the Wiener-Kintchine theorem
to the case of nonstationary processes. It is also found to be given by the
Fourier transform of Loeve's generalized power spectral density of a second
order nonstationary process and, in addition, is found to represent the special
cases of the stationary process, the locally stationary process, and the deter-
ministic process. In addition to the GIPS of the ensemble, i.e., the mean of
the spectrum ensemble, the covariance of the spectrum ensemble is also required
and a simple bound is shown to be given by the double Fourier transform of the
covariance of the signal covariance estimate. Since these expressions for the
GIPS and the spectral covariance require that the signal covariance be known,
they are primarily of theoretical interest. However, it is also shown that
estimates of the GIPS by means of an ensemble of Sliding FFT calculations will
be a smoothed version of the correct GIPS and the covariance of the ensemble
is a smoothed version of the correct covariance.

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